# Optimal Boolean Functions

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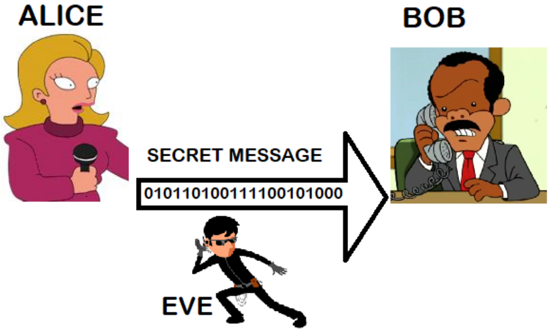
Communicate a secret message



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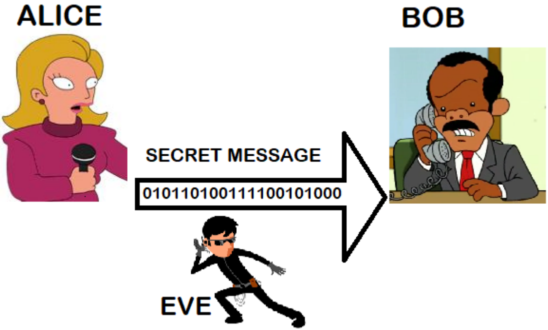


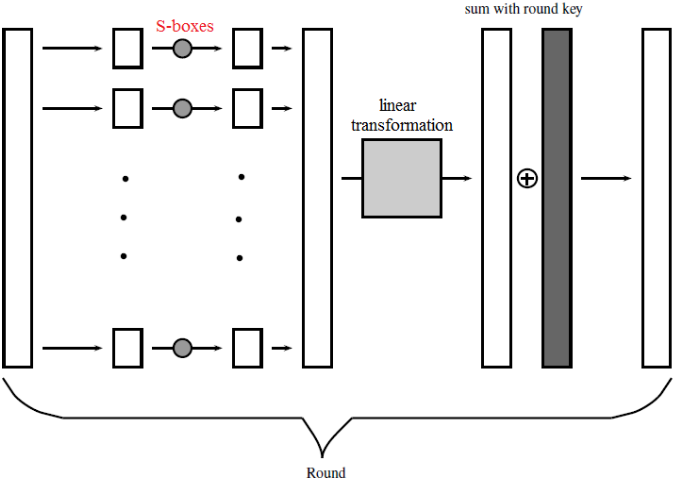

Cipher: $\mathcal{M}$ set of possible messages

$k \in \mathcal{K}$ key-space

$\varphi_k : \mathcal{M} \to \mathcal{M}$ encryption function

# Block ciphers

Example of translation based cipher



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$$F(x_1, \ldots, x_n) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix}, \ f_i : \mathbb{F}_2^n \to \mathbb{F}_2$$

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If $n = m$ an equivalent representation (univariate polynomial)

$$\boxed{F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}} \qquad F(x) = \sum_{i=0}^{2^n-1} c_i x^i, \; c_i \in \mathbb{F}_{2^n}.$$

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- differential attack
- linear cryptanalysis

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  how differences in an input can affect the resulting difference at the output.

$$\begin{array}{ccccc} x & \rightarrow & \left[\begin{array}{c} \\ F \\ \\ \end{array}\right] & \rightarrow & y \\ x+a & \rightarrow & & \rightarrow & y+b \end{array}$$

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  - best resistance when $\delta = 2^{n-m}$: PERFECT NONLINEAR (PN) $n$ even and $m \leq \frac{n}{2}$
  - if $n = m$ smallest $\delta = 2$: ALMOST PERFECT NONLINEAR (APN)

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$$d_H(f,g) = |\{x \in \mathbb{F}_2^n : f(x) \neq g(x)\}| \text{ (Hamming distance)}$$

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  - best resistance when NL is maximum: BENT
    $n$ even and $m \leq \frac{n}{2}$
  - if $n = m$: $NL(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}}$ ALMOST BENT (AB)

# CCZ-equivalence relation

Most general equivalence relation known that preserves $\delta$ and NL

Graph of a function $F$: $\Gamma_F = \{(x, F(x)) : x \in \mathbb{F}_2^n\}$

$F_1$ and $F_2$ are CCZ-equivalent if $\mathcal{L}(\Gamma_{F_1}) = \Gamma_{F_2}$, for an affine permutation $\mathcal{L}$.

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Other applications of APN and AB functions:

- coding theory
- sequence design
- combinatorial analysis

# On APN and AB functions $\boxed{F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}}$

- classification of APN, AB f. is an hard open problem
- complete classification known only for $n \leq 5$
- few infinite classes of APN and AB functions known
  - 6 infinite families of power APN f. (4 are also AB)
    (for example $x^{2^i+1}$ with $\gcd(i, n)=1$)
  - 11 infinite families of quadratic APN f. (4 are also AB)
- even for small $n$ there are too many vectorial Boolean functions to just use a purely computer search
- just one APN permutation is known in even dimension

We have to come up with new methods to construct new optimal functions and to analyse them

- combination of theoretic results and computational insights to find new families
- studying equivalence relations between already known functions
- finding new invariant of the CCZ-equivalence to easily prove CCZ-inequivalent functions
- finding more general equivalence relations that preserve optimal properties

# Example

- many known APN functions in small dimensions are of the form $F(x) = L_1(x^3) + L_2(x^9)$, with $L_1, L_2$ linear functions:
  - $x^3$ and $x^3 + Tr(x^9)$ are infinite families of APN functions
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- theoretical properties and restrictions on $L_1$ and $L_2$ for such function to be APN in $\mathbb{F}_{2^n}$:
  • if $F(x)$ is APN for an even $n$ then $F(a) \neq 0$ for any $a \neq 0$;
  • if $F(x)$ is APN for $n = 6m$ then $L_1(a^3\beta) \neq 0$ for any $a \neq 0$ and $\beta \in \mathbb{F}_{2^3}^*$ with $\mathrm{Tr}_3(\beta) = \beta^{2^2} + \beta^2 + \beta = 0$;

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  • $n = 8$ $x^9 + L(x^3)$ w. $L(x) = \alpha x^4 + \alpha^{-1}x^2 + \alpha^{-2}x$ is APN
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- CCZ-equivalent to an already known APN function $x^3$

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In many situations we want the cipher to be invertible

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Dream goal:

- find other APN permutations in even dimension
- find a family of APN permutations in even dimension

# Takk